

Generalized thermoelastic diffusive waves in heat conducting materials

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Abstract

Keeping in view the applications of diffusion processes in geophysics and electronics industry, the aim of the present paper is to give a detail account of the plane harmonic generalized thermoelastic diffusive waves in heat conducting solids. According to the characteristic equation, three longitudinal waves namely, elastodiffusive (ED), mass diffusion (MD-mode) and thermodiffusive (TD-mode), can propagate in such solids in addition to transverse waves. The transverse waves get decoupled from rest of the fields and hence remain unaffected due to temperature change and mass diffusion effects. These waves travel without attenuation and dispersion. The other generalized thermoelastic diffusive waves are significantly influenced by the interacting fields and hence suffer both attenuation and dispersion. At low frequency mass diffusion and thermal waves do not exist but at high-frequency limits these waves propagate with infinite velocity being diffusive in character. Moreover, in the low-frequency regions, the disturbance is mainly dominant by mechanical process of transportation of energy and at high-frequency regions it is significantly dominated by a close to diffusive process (heat conduction or mass diffusion). Therefore, at low-frequency limits the waves like modes are identifiable with small amplitude waves in elastic materials that do not conduct heat. The general complex characteristic equation is solved by using irreducible case of Cardano's method with the help of DeMoivre's theorem in order to obtain phase speeds, attenuation coefficients and specific loss factor of energy dissipation of various modes. The propagation of waves in case of non-heat conducting solids is also discussed. Finally, the numerical solution is carried out for copper (solvent) and zinc (solute) materials and the obtained phase velocities, attenuation coefficients and specific loss factor of various thermoelastic diffusive waves are presented graphically.

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1. Introduction

During the last three decades, non-classical theories of thermoelasticity so-called 'Generalized thermoelasticity' have been developed in order to remove the paradox of physically impossible phenomenon of infinite velocity of thermal signals in the conventional coupled thermoelasticity. For example, Lord and Shulman [1] formulated a generalized theory of thermoelasticity with one thermal relaxation time which involves a hyperbolic equation of heat transportation by incorporating a flux-rate term into the Fourier's law of heat conduction. Green and Lindsay [2] developed a temperature-rate-dependent thermoelasticity that

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includes two thermal relaxation times and does not violate the classical Fourier law of heat conduction, when the body under consideration has a center of symmetry. The Lord and Shulman [1] theory of generalized thermoelasticity was further extended to homogeneous anisotropic heat conducting materials; by Dhaliwal and Sherief [3]. All these theories predict finite speed of heat propagation. Chandrashekhariah [4] referred to this wave like thermal disturbance as “second sound”. A survey article of various representative theories in the range of generalized thermoelasticity was brought out by Hetnarski and Ignaczak [5].

The thermo diffusion in elastic solids is due to coupling of the fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with environment. Nowacki [6–9] developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Dudziak and Kowalski [10] and Olesiak and Pyryev [11], respectively, discussed the theory of thermo diffusion and coupled quasi-stationary problems of thermal diffusion for an elastic cylinder. They studied the influence of cross effects arising from the coupling of the fields of temperature, mass diffusion and strain due to which the thermal excitation results in additional mass concentration and that generates additional fields of temperature.

The recent development of generalized theory of thermoelastic diffusion by Sherief et al. [12] allows the finite speed of propagation of waves and it provides a chance to study wave propagation in such interesting media. They derived governing equations for generalized thermo diffusion in elastic solids and also proved variational principles and reciprocity theorems for these equations. The uniqueness of solution for these equations under suitable conditions is also established. The propagation of plane harmonic waves in generalized thermoelastic solids have been discussed by many authors [13–16].

Singh [17] investigated the reflection of P and SV waves from the free surface of elastic solids with generalized thermodiffusion. According to Sherief et al. [12] diffusion can be defined as the random walk of an ensemble of particles from regions of high concentration to that of low concentration. Nowadays there is a great deal of interest in the study of this phenomenon due to its applications in geophysics and electronic industry. In integrated circuit fabrication diffusion is used to introduce “dopants” in controlled amounts into the semiconductor substance. In particular, diffusion is used to form the base and emitter in bipolar transistors, integrated resistors, and the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of the applications, the concentration is calculated using what is known as Fick’s law. This is a simple law which does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of the temperature on this interaction.

Keeping in view, the above applications of thermodiffusive processes, the propagation of generalized thermoelastic diffusive waves has been investigated in this paper. The phase velocities and attenuation coefficients of various possible modes of wave propagation have been computed by using irreducible case of Cardano’s method with the help of DeMoivre’s theorem from the secular equations. The propagation of waves in non-heat conducting materials has also been discussed. The analytical results have also been computed numerically and represented graphically for illustration of various physical phenomenon inherited by such solids.

2. Basic equations

The basic governing equations for linear thermodiffusive interactions in homogeneous isotropic elastic solids are [12],

i. Strain–displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3. \quad (1)$$

ii. Stress–strain–temperature–concentration relations for such materials:

$$\begin{aligned} \sigma_{ij} &= \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} - \beta_1 T \delta_{ij} - \beta_2 C \delta_{ij}, \\ \rho T_0 S &= \rho C_e T + \beta_1 T_0 e_{kk} + a T_0 C, \\ P &= -\beta_2 e_{kk} + b C - a T, \quad i, j, k = 1, 2, 3. \end{aligned} \quad (2)$$

These are also known as constitutive relations.

iii. Equations of motion

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} - \beta_1 T_{,i} - \beta_2 C_{,i} + \rho F_i = \rho \ddot{u}_i, \quad i, j = 1, 2, 3 \tag{3}$$

iv. Equation of heat conduction

$$KT_{,ii} - \rho C_e(\dot{T} + t_0 \ddot{T}) = \beta_1 T_0(\dot{e} + t_0 \ddot{e}) + aT_0(\dot{C} + t_0 \ddot{C}), \quad i = 1, 2, 3. \tag{4}$$

v. Equation of mass diffusion

$$C_{,ii} - \frac{1}{Db}(\dot{C} + t_1 \ddot{C}) = \frac{\beta_2}{b} e_{,ii} + \frac{a}{b} T_{,ii}, \quad i = 1, 2, 3, \tag{5}$$

where $\beta_1 = (3\lambda + 2\mu)\alpha_T$, $\beta_2 = (3\lambda + 2\mu)\alpha_C$, e , dilatation; λ and μ are Lames parameters; α_T , coefficients of linear thermal expansion and α_C , coefficients of linear diffusion expansion; ρ , density; C_e , specific heat; a , thermodiffusive constants; b , diffusive constant; K , thermal conductivity; P , chemical potential; C , concentration; T , temperature change; t_0, t_1 , thermal relaxation time parameters and u_i , displacement components. We assume that the material parameters satisfy the inequalities

$$T_0 > 0, \quad t_0 > 0, \quad t_1 > 0, \quad \mu > 0, \quad \lambda > 0, \quad K > 0, \quad D > 0, \quad C_e > 0, \quad \rho > 0. \tag{6}$$

3. Formulation and solution of the problem

We consider a homogeneous isotropic thermodiffusive elastic solid at uniform temperature T_0 and initial concentration C_0 , in the undisturbed state. The basic field equations (3)–(5) of linear generalized thermodiffusion, in the absence of body forces and heat sources, become

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla \nabla \cdot \vec{u} - \beta_1 \nabla T - \beta_2 \nabla C = \rho \ddot{\vec{u}}, \tag{7}$$

$$K \nabla^2 T - \rho C_e (\dot{T} + t_0 \ddot{T}) = \beta_1 T_0 \nabla \cdot (\dot{\vec{u}} + t_0 \ddot{\vec{u}}) + a T_0 (\dot{C} + t_0 \ddot{C}), \tag{8}$$

$$\nabla^2 C - \frac{1}{Db} (\dot{C} + t_1 \ddot{C}) = \frac{\beta_2}{b} \nabla^2 (\nabla \cdot \vec{u}) + \frac{a}{b} \nabla^2 T, \tag{9}$$

where $\vec{u}(x_1, x_2, x_3, t) = (u_1, u_2, u_3)$ and $T(x_1, x_2, x_3, t)$ are, respectively, the displacement vector and temperature change. We define the quantities

$$\begin{aligned} x'_i &= \frac{\omega^* x_i}{c_L}, \quad t' = \omega^* t, \quad u'_i = \frac{\rho \omega^* c_L u_i}{\beta_1 T_0}, \quad T' = \frac{T}{T_0}, \quad C' = \frac{C}{C_0}, \quad t'_0 = \omega^* t_0, \quad t'_1 = \omega^* t_1, \\ \omega^* &= \frac{C_e(\lambda + 2\mu)}{K}, \quad \varepsilon_T = \frac{T_0 \beta_1^2}{\rho C_e(\lambda + 2\mu)}, \\ \delta^2 &= \frac{c_S^2}{c_L^2}, \quad \bar{\beta} = \frac{\beta_2 C_0}{\beta_1 T_0}, \quad \bar{b} = \frac{a T_0}{b C_0}, \quad \bar{a} = \frac{a C_0}{\rho C_e}, \end{aligned} \tag{10}$$

$$\varpi_b = \frac{c_L^2}{\omega^* D b}, \quad \varepsilon_c = \frac{\beta_1 \beta_2 T_0}{C_0 b (\lambda + 2\mu)}, \quad c' = \frac{c}{c_L}, \quad k' = \frac{k c_L}{\omega^*}, \quad \omega' = \frac{\omega}{\omega^*}, \quad c_L^2 = \frac{(\lambda + 2\mu)}{\rho}, \quad c_S^2 = \frac{\mu}{\rho}.$$

Upon introducing quantities (10) in Eqs. (7)–(9), we obtain

$$\delta^2 \nabla^2 \vec{u} + (1 - \delta^2) \nabla \nabla \cdot \vec{u} - \nabla T - \bar{\beta} \nabla C = \ddot{\vec{u}}, \tag{11}$$

$$\nabla^2 T - (\dot{T} + t_0 \ddot{T}) - \varepsilon_T \nabla \cdot (\dot{\vec{u}} + t_0 \ddot{\vec{u}}) + \bar{a} (\dot{C} + t_0 \ddot{C}) = 0, \tag{12}$$

$$\nabla^2 C - \varpi_b(\dot{C} + t_1 \ddot{C}) - \varepsilon_c \nabla^2(\nabla \cdot \vec{u}) - \bar{b} \nabla^2 T = 0. \tag{13}$$

Here dashes have been omitted for convenience. A plane displacement wave of harmonic time dependence propagating with non-dimensional phase velocity c in a direction defined by the propagation vector \vec{p} is represented as

$$\vec{u} = A_1 \vec{d} \exp \{ ik(\vec{r} \cdot \vec{p} - ct) \}, \tag{14}$$

where k , \vec{r} and \vec{d} are, respectively, the wave number, the position vector, and a unit vector defining the direction of motion. The deformation affects the thermal state and concentration of the medium. Therefore, a displacement wave is accompanied by thermal and mass diffusion fields, which are scalar quantities, and may be taken of the similar form [16] as

$$T = A_2 \exp \{ ik(\vec{r} \cdot \vec{p} - ct) \}, \tag{15.1}$$

$$C = A_3 \exp \{ ik(\vec{r} \cdot \vec{p} - ct) \}. \tag{15.2}$$

Substitution of solutions (14) to (15) in Eqs. (11)–(13) leads to

$$(\delta^2 - c^2)A_1 \vec{d} + (1 - \delta^2)(\vec{p} \cdot \vec{d})\vec{p}A_1 + ik^{-1}\vec{p}A_2 + ik^{-1}\bar{\beta}\vec{p}A_3 = 0, \tag{16}$$

$$ik\varepsilon_T \tau_0 c^2 (\vec{p} \cdot \vec{d})A_1 - (1 - \tau_0 c^2)A_2 + \bar{a} \tau_0 c^2 A_3 = 0, \tag{17}$$

$$ik\varepsilon_c (\vec{p} \cdot \vec{d})A_1 + \vec{b}A_2 - (1 - \bar{\omega}_b \tau_1 c^2)A_3 = 0, \tag{18}$$

where $\tau_0 = t_0 + i\omega^{-1}$, $\tau_1 = t_1 + i\omega^{-1}$.

The system of equations (16)–(18) has a non-trivial solution if the determinant of the coefficients of A_1 , A_2 and A_3 vanishes. This leads to the secular equation

$$(\delta^2 - c^2)\vec{d} + (1 - \delta^2 + \varepsilon_a)(\vec{p} \cdot \vec{d})\vec{p} - \frac{[\varepsilon_a + (\varepsilon_T - \varepsilon_a)\tau_0 c^2][1 - \varpi_b \tau_1 c^2 + \bar{\beta}\bar{b}]}{(1 - \tau_0 c^2)(1 - \bar{\omega}_b \tau_1 c^2) - \bar{a}\bar{b}\tau_0 c^2} (\vec{p} \cdot \vec{d})\vec{p} = 0, \tag{19}$$

where $\varepsilon_a = \varepsilon_c/\bar{b}$. Eq. (19) is a vector equation consisting of a set of three equations which must be solved for unknown vector \vec{d} . We adopt the approach of Ahmad and Khan [16] to solve Eq. (19). If we take $\vec{d} = (d_1, d_2, d_3)$ and $\vec{p} = (p_1, p_2, p_3)$, Eq. (19) is equivalent to the following system of equations:

$$(a_{ij})d_j = 0, \tag{20}$$

where the matrix (a_{ij}) is given by

$$(a_{ij}) = \begin{bmatrix} \delta^2 - c^2 + Mp_1^2 & Mp_1 p_2 & Mp_1 p_3 \\ Mp_1 p_2 & \delta^2 - c^2 + Mp_2^2 & Mp_2 p_3 \\ Mp_1 p_3 & Mp_2 p_3 & \delta^2 - c^2 + Mp_3^2 \end{bmatrix}, \tag{21}$$

$$M = 1 - \delta^2 - \varepsilon_a - T, \quad T = \frac{[\varepsilon_a + (\varepsilon_T - \varepsilon_a)\tau_0 c^2][1 - \tau_1 \varpi_b c^2 + \bar{\beta}\bar{b}]}{[1 - \tau_0 c^2][1 - \varpi_b \tau_1 c^2] - \bar{a}\bar{b}\tau_0 c^2}. \tag{22}$$

For a non-trivial solution of Eq. (20) the determinant of matrix (a_{ij}) must be zero. This gives us a characteristic equation which can be solved for c^2 . In the absence of mass concentration ($a = 0 = \beta_2$) and thermo-mechanical coupling ($\varepsilon_T = 0$), the matrix (21) is symmetric and the three eigenvalues are real, corresponding to three elastic waves propagating in any fixed direction. However, in the presence of thermal coupling ($\varepsilon_T \neq 0$) and absence of mass concentration ($a = 0 = \beta_2$), the matrix (a_{ij}) is neither symmetric nor Hermitian. The characteristic equation in this case, after rationalization and simplification, is of fourth degree in c^2 having in general, complex roots. Because the thermal coupling is usually small, three of the roots will have small imaginary parts and have real parts approximately equal to the corresponding eigenvalues of (a_{ij}) with $\varepsilon_T = 0$. These three roots give three attenuated elastic waves and the fourth root corresponds to the thermal wave which has a large attenuation coefficient. When both mass concentration and thermoelastic coupling are present ($\varepsilon_T \neq 0$, $a \neq 0$, $\beta_2 \neq 0$), the matrix (a_{ij}) is not Hermitian (symmetric) again. The characteristic equation

$|a_{ij}| = 0$, after rationalization and simplification, is of fifth degree in c^2 having, in general, complex roots. Here four of the five roots will be associated with attenuated elastic and thermal waves. The fifth root will be associated to the mass diffusion wave (*MD-mode*) having large attenuation coefficient.

4. Longitudinal and transverse waves

In this section we consider two situations of polarization vector \vec{d} and wave normal \vec{p} namely, either $\vec{d} = \pm\vec{p}$, or $\vec{p} \cdot \vec{d} = 0$. The former corresponds to longitudinal motion and latter to transverse one in the generalized thermoelastic solid with voids.

4.1. Transverse waves

If $\vec{d} \neq \pm\vec{p}$, then $\vec{p} \cdot \vec{d} = 0$ and Eq. (19) implies that

$$c = \pm\delta. \tag{23}$$

This Eq. (23) defines transverse wave which do not interact with temperature and mass diffusion fields and travels with non-dimensional velocity δ without dispersion, attenuation and dissipation.

4.2. Longitudinal waves

If $\vec{d} = \pm\vec{p}$ then $\vec{p} \cdot \vec{d} \neq 0$ and Eq. (19) in this case leads to

$$(1 + \varepsilon_a - c^2) - \frac{[\varepsilon_a + \tau_0 c^2(\varepsilon_T - \varepsilon_a)] [1 + \beta\bar{b} - \varpi_b \tau_1 c^2]}{(1 - \tau_0 c^2)(1 - \tau_1 \varpi_b c^2) - \bar{a}\bar{b}\tau_0 c^2} = 0. \tag{24}$$

This complicated equation (24) shows that the phase velocity depends on ω or k , meaning that thermoelastic diffusive waves are dispersive. Because the solution of Eq. (24) for c is generally complex valued, so generalized thermoelastic diffusive waves also suffer attenuation. In low- and high-frequency regimes, characterized by $\omega \ll 1$ and $\omega \gg 1$, the wave like modes are determined by isentropic (constant entropy) and isothermal material parameters, respectively. A motivating argument runs as follow:

Consider the transport of energy over a distance d in the direction defined by $\vec{p} = (1, 0, 0)$. Representative times for this process are $t_m = d((\lambda + 2\mu)/\rho)^{-1/2}$, in case the energy is mechanical and transported through a displacement wave, and $t_T = d^2(K/(\rho C_e |\tau_0^*)))^{-1}$, $\tau_0^* = 1 - i\omega t_0$ when the energy is thermal and the operational mechanism is heat conduction. In case the operative mechanism is mass diffusion, the representative time $t_M = d^2(Db/|\tau_1^*|)^{-1}$, $\tau_1^* = 1 - i\omega t_1$. The frequency of the wave is $\omega = t_m^{-1}$, hence $|t_m/t_T| = \omega[1 - (t_0^2\omega^2/2) + o(t_0^4)]$. We deduce that $t_m \ll t_T$ when $\omega \ll 1$, indicating that at low frequencies mechanical energy transfer is more effective than thermal conduction and conditions locally are therefore, nearly isentropic. Whereas, at the opposite extreme, $\omega \gg 1$, thermal energy transfer is more predominant process, and the prevailing conditions are nearly isothermal. Thus at the low frequency limits, the wavelike modes are identifiable with the small amplitude waves in an elastic material that does not conduct heat and may be regarded as inherent in the classical elastodynamics derived strictly from mechanical principles. Similarly we have $|t_m/t_M| = \omega[1 - t_1^2\omega^2/2 + o(t_1^4)]$. We again deduce that $t_m \ll t_M$ at low frequency $\omega \ll 1$ and mechanical energy transfer is more effective than mass diffusion, while at high frequency $\omega \gg 1$ mass diffusion is more predominant process.

We now inspect the low and high-frequency behavior of waves. Eq. (24) can also be rewritten as

$$1 + \varepsilon_a - c^2 - \frac{[(i + t_0\omega)c^2(\varepsilon_T - \varepsilon_a) + \varepsilon_a\omega] [(1 + \beta\bar{b})\omega - \varpi_b c^2(t_1\omega + i)]}{[\omega - (t_0\omega + i)c^2][\omega - \varpi_b(t_1\omega + i)c^2] - \bar{a}\bar{b}\omega(t_0\omega + i)c^2} = 0. \tag{25}$$

At low frequency ($\omega \rightarrow 0$), the secular equation (25) provides us

$$c = 0, 0, \sqrt{1 + \varepsilon_T}. \tag{26}$$

The modes associated with the first two values of c correspond to thermodiffusive wave (*TD-mode*) and mass diffusion wave (*MD-mode*) and the third value of c is associated with elastodiffusive (*ED*) wave. Obviously,

there is no damping in either of these modes in this limiting case. Thus the phase velocity of elastic (ED) wave is slightly increased due to thermomechanical coupling effects at adiabatic conditions in this limiting case as compared to its value at isothermal conditions in elastokinetics.

At high frequency ($\omega \rightarrow 0$), the secular equation can be written from Eq. (24) in a straight forward manner by just replacing τ_0 and τ_1 with t_0 and t_1 , respectively. We get

$$(1 + \varepsilon_a - c^2) - \frac{[\varepsilon_a + (\varepsilon_T - \varepsilon_a)t_0c^2][1 + \bar{\beta}\bar{b} - \varpi_b t_1 c^2]}{(1 - t_0c^2)(1 - t_1\varpi_b c^2) - \bar{a}\bar{b}t_0c^2} = 0. \tag{27}$$

This reduced frequency equation (27) provides us three real values of c^2 for non-zero t_0 and t_1 which correspond to three modes of wave propagation namely, thermodiffusive wave (TD-mode), mass diffusion (MD-mode) wave and elastodiffusive (ED) wave. Clearly all the waves are influenced and affected by thermal relaxation and mass diffusion effects. In the absence of thermal relaxation time ($t_0 = 0 = t_1$), the secular equation (27) leads to

$$c = \sqrt{1 - \varepsilon_c \bar{\beta}} \tag{28.1}$$

for longitudinal elastic wave and other two modes have infinite speeds of propagation being diffusive in character. Clearly, the velocity of elastic wave is influenced and gets reduced due to mass concentration in the material.

In the absence of mass concentration ($a = 0 = \beta_2 \Rightarrow \varepsilon_c, \varepsilon_a = 0$), Eq. (27) provides us

$$c = \frac{\pm 1}{\sqrt{t_1 \varpi_b}}, \quad c = \pm \sqrt{\frac{((1 + \varepsilon_T + (1/t_0)) \pm \sqrt{(1 + \varepsilon_T + (1/t_0))^2 - (4/t_0)})}{2}}. \tag{28.2}$$

Further in the absence of thermo mechanical coupling ($\varepsilon_T = 0$), Eq. (28.2) leads to

$$c = \frac{\pm 1}{\sqrt{t_1 \varpi_b}}, \quad c = \pm 1, \quad c = \frac{\pm 1}{\sqrt{t_0}}. \tag{28.3}$$

For $t_0 = 0 = t_1$, Eq. (28.3) provides us $c = 1, \infty, \infty$, which implies that in this limiting case the longitudinal elastic wave travels with dimensional velocity equal to isothermal value $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ as in elastokinetics and the other two waves have infinite velocity of propagation being diffusive in character.

In the general case, the secular equation (24), after lengthy but straight forward algebraic reduction and manipulation, can be rewritten as

$$\prod_{i=1}^3 (c^2 - a_i^2) = 0, \tag{29}$$

where $a_i^2, i = 1, 2, 3$ are the roots of the complex cubic equation

$$\xi^3 - L\xi^2 + M\xi - N = 0. \tag{30}$$

Here

$$L = \{ \tau_0(1 + \bar{a}\bar{b}) + \varpi_b \tau_1 [1 + \tau_0(1 + \varepsilon_T)] \} / \varpi_b \tau_1 \tau_0, \tag{31.1}$$

$$M = \{ 1 + \tau_1 \varpi_b + \tau_0 [(1 + \bar{a}\bar{b})(1 + \varepsilon_a) + (1 + \bar{\beta}\bar{b})(\varepsilon_t - \varepsilon_a)] \} / \tau_0 \tau_1 \varpi_b, \tag{31.2}$$

$$N = (1 - \varepsilon_c \bar{\beta}) / \tau_0 \tau_1 \varpi_b. \tag{31.3}$$

Eq. (30) is a cubic equation with complex coefficients and there is no general arithmetic or algebraic method of finding the exact value of the cube roots of imaginary quantities. Therefore, this equation will be solved by using irreducible case of Cardan’s method with the help of DeMoivre’s theorem. Thus the complex roots $a_j^2 = \xi_j, j = 1, 2, 3$ become known and consequently we have

$$c_j = \pm a_j, \quad j = 1, 2, 3. \tag{32}$$

We define

$$c^{-1} = V^{-1} + i\omega^{-1}Q, \quad k = R + iQ, \quad R = \frac{\omega}{V}. \tag{33}$$

Here V and Q are real, the exponent in the plane wave solutions (14)–(15) becomes $-Q(\vec{r} \cdot \vec{p}) - i\omega(V^{-1}\vec{r} \cdot \vec{p} - t)$. This shows that V is the propagation speed and Q is the attenuation coefficient of the waves. Upon using representation (33) in Eq. (32), we obtain

$$V_j = \frac{1}{\text{Re}(a_j)}, \quad Q_j = \omega \text{Im}(a_j), \quad j = 1, 2, 3. \tag{34}$$

In the absence of mass concentration ($a = 0 = \beta_2 \Rightarrow \varepsilon_c, \varepsilon_a \rightarrow 0$), Eq. (29) provides us

$$c = \frac{\pm 1}{\sqrt{t_1 \omega b}}, \quad c = \pm \sqrt{\frac{(1 + \varepsilon_T + (1/\tau_0)) \pm \sqrt{(1 + \varepsilon_T + (1/\tau_0))^2 - (4/\tau_0)}}{2}}. \tag{35.1}$$

Further in the absence of thermo mechanical coupling ($\varepsilon_T = 0$), Eq. (35.1) leads to

$$c = \frac{\pm 1}{\sqrt{t_1 \omega b}}, \quad c = \pm 1, \quad c = \frac{\pm 1}{\sqrt{\tau_0}}. \tag{35.2}$$

Eq. (35.1) for $\omega \rightarrow 0$ leads to $c = 0, 0, \sqrt{1 + \varepsilon_T}$ which agrees with Eq. (27).

Upon using the representation (33) and $t_0, t_1 \ll 1$, Eq. (35.2) provides us

$$V_1 = 1, \quad V_2 = \frac{\sec \theta}{\sqrt{t_0}}, \quad V_3 = \frac{\sec \theta}{\sqrt{t_1 \omega b}}, \quad Q_1 = 0.0, \quad Q_2 = \omega \sqrt{t_0} \sin \theta, \quad Q_3 = \omega \sqrt{t_1 \omega b} \sin \phi,$$

where

$$\theta = \frac{\tan^{-1}(1/t_0 \omega)}{2}, \quad \phi = \frac{\tan^{-1}(1/t_1 \omega)}{2}.$$

These relations agree with the consequent results of Eq. (28.3) for $t_0 = 0 = t_1$.

The solution of limiting secular equation (27) at high frequency limits can also be obtained in a similar manner and the corresponding phase velocities and attenuation coefficient can be derived and calculated from Eq. (34). The results in the context of conventional coupled thermoelasticity can be obtained from the above analysis by setting $t_0 = 0 = t_1$. The values of phase velocity and attenuation coefficients in dimensional form can be obtained with the straight forward use of quantities (10).

5. Non-heat conducting solids

If the solids is not capable of conducting heat ($K \rightarrow 0$), then Eq. (8) implies that

$$T = -\frac{\beta_1 T_0}{\rho C_e} \nabla \cdot \vec{u} - \frac{a T_0}{\rho C_e} C. \tag{36}$$

Upon using Eq. (36) in Eqs. (7) and (9), we get

$$\delta^2 \nabla^2 \vec{u} + (1 + \varepsilon_T - \delta^2) \nabla \nabla \cdot \vec{u} - \frac{a \varepsilon_T}{\beta_1} \nabla c = \frac{1}{c^2} \ddot{\vec{u}}, \tag{37}$$

$$\nabla^2 C - \frac{1}{Db(1 + \varepsilon_b)} (\dot{C} + t_1 \ddot{C}) = \frac{\beta_1}{a} (\vec{a}^* \vec{\beta}^* - \varepsilon_b) \nabla^2 \nabla \cdot \vec{u}, \tag{38}$$

where $\vec{a}^* = a/b$, $\varepsilon_b = a^2 T_0 / \rho C_e b$, $\vec{\beta}^* = \beta_2 / \beta_1$.

Eqs. (37) and (38) with the help of Eqs. (10), (14) and (15) leads to

$$(\delta^2 - c^2)\vec{d} + (1 + \varepsilon_T - \delta^2)(\vec{p} \cdot \vec{d})\vec{p} - \frac{\varepsilon_T(\vec{a}^*\vec{\beta}^* - \varepsilon_b)}{1 - (c_L^2\tau_1c^2)/(Db(1 + \varepsilon_b))}\vec{p}(\vec{p} \cdot \vec{d}) = 0, \tag{39}$$

where $\tau_1 = t_1 + i\omega^{-1}$ and c is the non-dimensional phase velocity.

The vector equation (39) consists of a set of three equations, which must be solved for unknown vector $\vec{d} = (d_1, d_2, d_3)$ for given $\vec{p} = (p_1, p_2, p_3)$. This system can be written in the matrix form as below:

$$(b_{ij})d_j = 0, \tag{40}$$

where the matrix (b_{ij}) is given by

$$(b_{ij}) = \begin{bmatrix} \delta^2 - c^2 + Sp_1^2 & Sp_1p_2 & Sp_1p_3 \\ Sp_1p_2 & \delta^2 - c^2 + Sp_2^2 & Sp_2p_3 \\ Sp_1p_3 & Sp_2p_3 & \delta^2 - c^2 + Sp_3^2 \end{bmatrix}, \tag{41}$$

$$S = 1 - \delta^2 + \varepsilon_T - T^*, \quad T^* = \frac{\varepsilon_T(\vec{a}^*\vec{\beta}^* - \varepsilon_b)}{1 - (c_L^2\tau_1c^2)/Db(1 + \varepsilon_b)}. \tag{42}$$

The system of equations (40) will have a non-trivial solution if the determinant of the matrix (b_{ij}) is zero. This gives us a characteristic equation that can be solved for c^2 . In the absence of mass concentration ($a = 0 = \beta_2$), the matrix (41) is Hermitian (symmetric) giving three real eigenvalues corresponding to three elastic waves propagating with adiabatic speeds. However, in the presence of mass concentration ($a \neq 0, \beta_2 \neq 0$) the matrix (b_{ij}) is not Hermitian and the characteristic equation in this case, after rationalization and simplification, is of fourth degree in c^2 having in general, complex roots. Here three of the four roots correspond to three elastic waves with small attenuation and the fourth one to the mass concentration wave (MD-mode) with large attenuation coefficient.

5.1. Transverse waves

In this case $\vec{p} \cdot \vec{d} = 0$ and hence Eq. (39) implies that $\delta^2 - c^2 = 0$ so that $c = \pm\delta$. This defines transverse wave at adiabatic conditions, which do not interact with temperature and mass concentration fields and hence travels with non-dimensional velocity δ without dispersion, attenuation and dissipation.

5.2. Longitudinal waves

For longitudinal waves, we have $\vec{p} = \pm\vec{d}, \vec{p} \cdot \vec{d} \neq 0$ and hence Eq. (39) implies that

$$(1 + \varepsilon_T - c^2) \left[1 - \frac{c_L^2\tau_1c^2}{Db(1 + \varepsilon_b)} \right] - \varepsilon_T(\vec{a}^*\vec{\beta}^* - \varepsilon_b) = 0. \tag{43}$$

In the absence of mass diffusion ($a = 0$), the above equation provides us

$$c = \pm\sqrt{1 + \varepsilon_T}, \quad \pm(\sqrt{Db/\tau_1})/c_L, \quad \tau_1 = t_1 + i\omega^{-1}. \tag{44}$$

The first value of phase velocity c corresponds to elastic modes at adiabatic conditions and the other to that of mass diffusion waves. Evidently second value of c being complex shows that mass diffusion modes are attenuated in space and first value is same as obtained in the low frequency limit ($\omega \rightarrow 0$) discussed in the previous section. Eq. (44) implies that

$$c = \pm\lambda_i, \quad i = 1, 2, \tag{45}$$

where $\lambda_i, i=1,2$ are given

$$\lambda_1^2 + \lambda_2^2 = (1 + \varepsilon_T) + \frac{Db(1 + \varepsilon_b)}{\tau_1c_L^2}, \quad \lambda_1^2\lambda_2^2 = \left\{ 1 + \varepsilon_T(1 - \vec{a}^*\vec{\beta}^* + \varepsilon_b) \right\} Db(1 + \varepsilon_b)/\tau_1c_L^2. \tag{46}$$

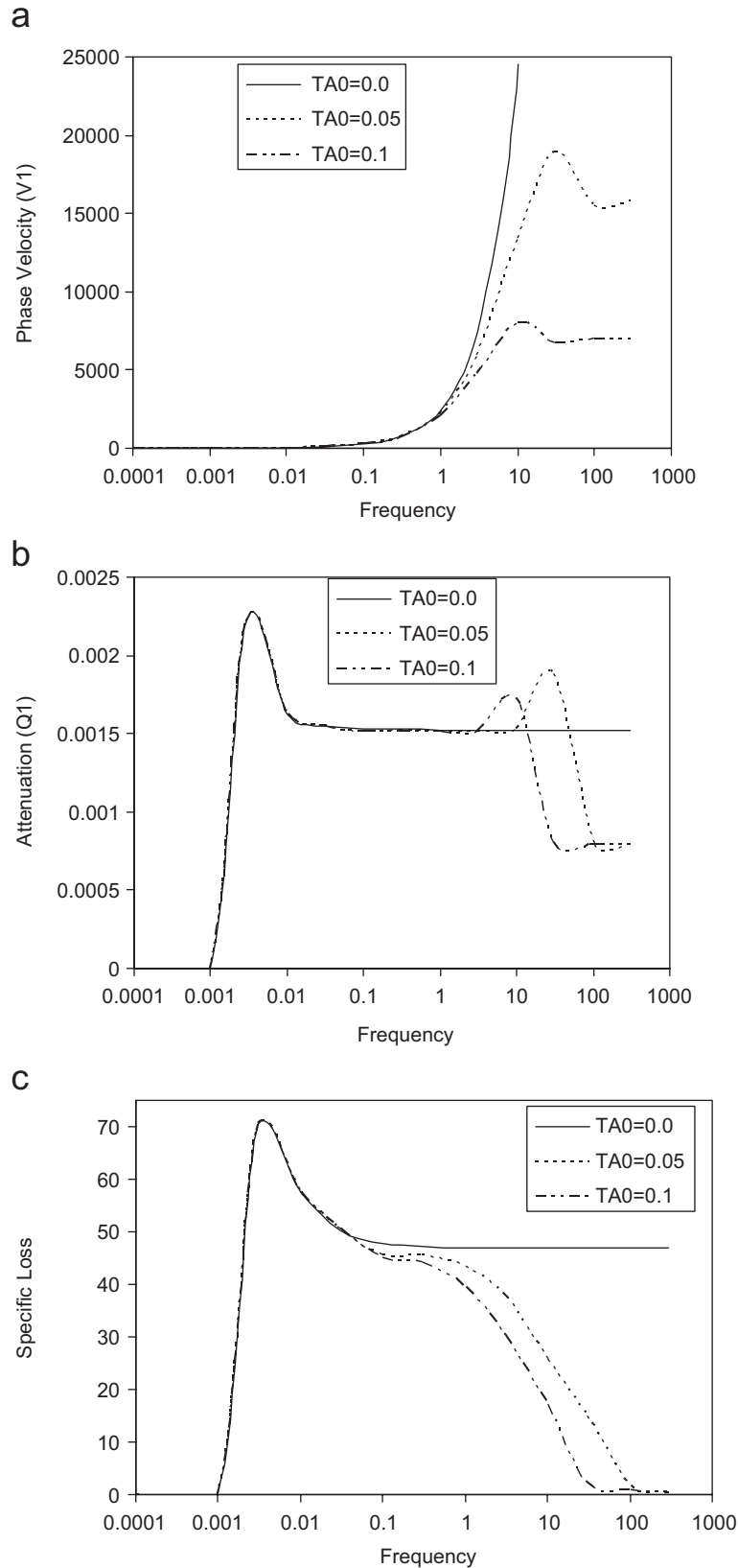
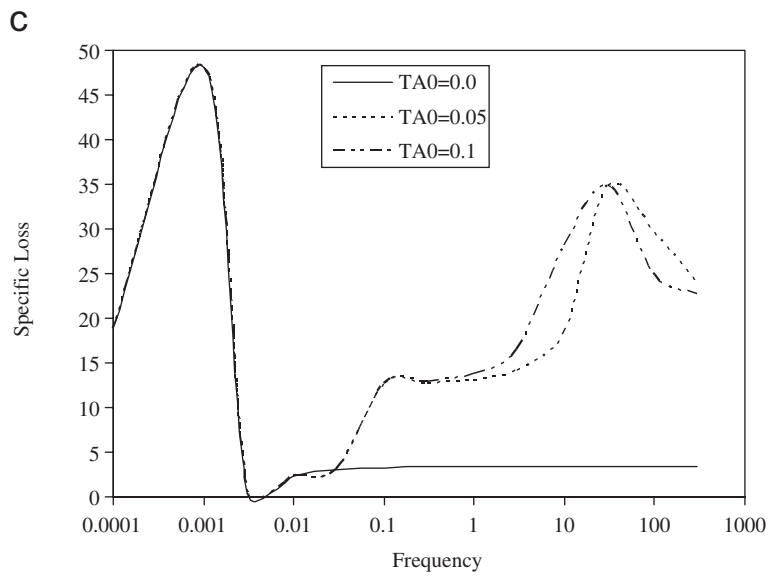
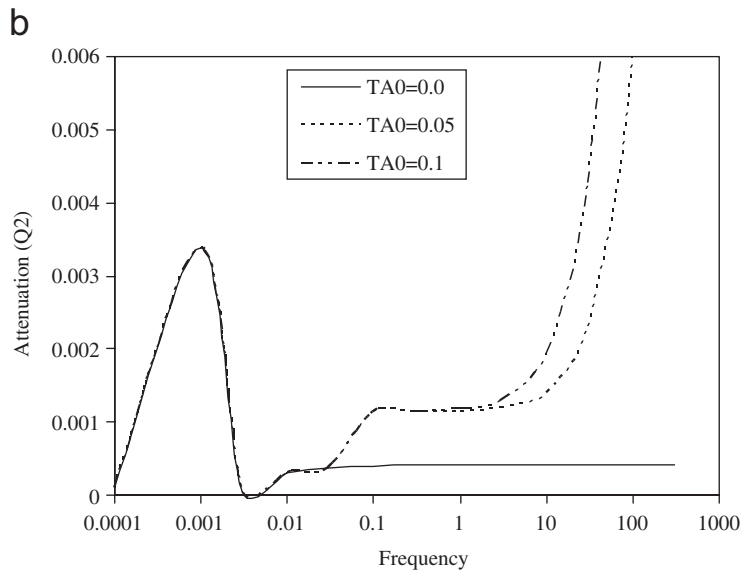
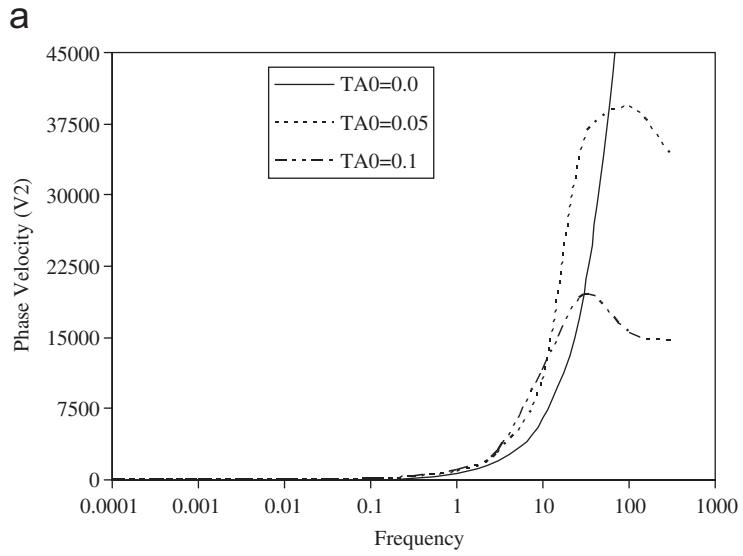


Fig. 1. (a) Variation of phase velocity of elastodiffusive (ED) waves w.r.t. frequency for different relaxation times. (b) Variation of attenuation coefficient of elastodiffusive (ED) waves w.r.t. frequency for different relaxation times. (c) Variation of specific loss factor of elastodiffusive (ED) waves w.r.t. frequency for different relaxation times.



Upon using representation (33), we obtain

$$V_i = 1/\text{Re}\left(\frac{1}{\lambda_i}\right), \quad Q_i = \omega \text{Im}\left(\frac{1}{\lambda_i}\right). \tag{47}$$

This determines the phase velocity and attenuation coefficients of possible modes of wave propagation one corresponds to elastic and other to mass diffusion. The results in the absence of mass diffusion relaxation effects can be obtained from the above analysis by setting $t_1 = 0$ in the relevant relations. The dimensional values of phase velocity and attenuation coefficients can be obtained with the help of quantities (10).

6. Specific loss

The specific loss is the energy dissipated in taking a specimen through a stress cycle, ΔW , to the elastic energy stored in the specimen when the strain is a maximum, W . The specific loss is the most direct method of defining internal friction for a material, Puri and Cowin [18]. For a sinusoidal plane wave of small amplitude, Kolsky [19, p. 106] shows that the specific loss $\Delta W/W$ equals 4π times the absolute value of the imaginary part of k to the real part of k , i.e. $\Delta W/W = 4\pi|\text{Im}(k)/\text{Re}(k)|$ where k is a complex number such that $\text{Im}(k) > 0$. Here

$$\frac{\Delta W}{W} = 4\pi \left| \frac{\text{Im}(k)}{\text{Re}(k)} \right| = 4\pi \left| \frac{V_i Q_i}{\omega} \right|. \tag{48}$$

7. Numerical results and discussion

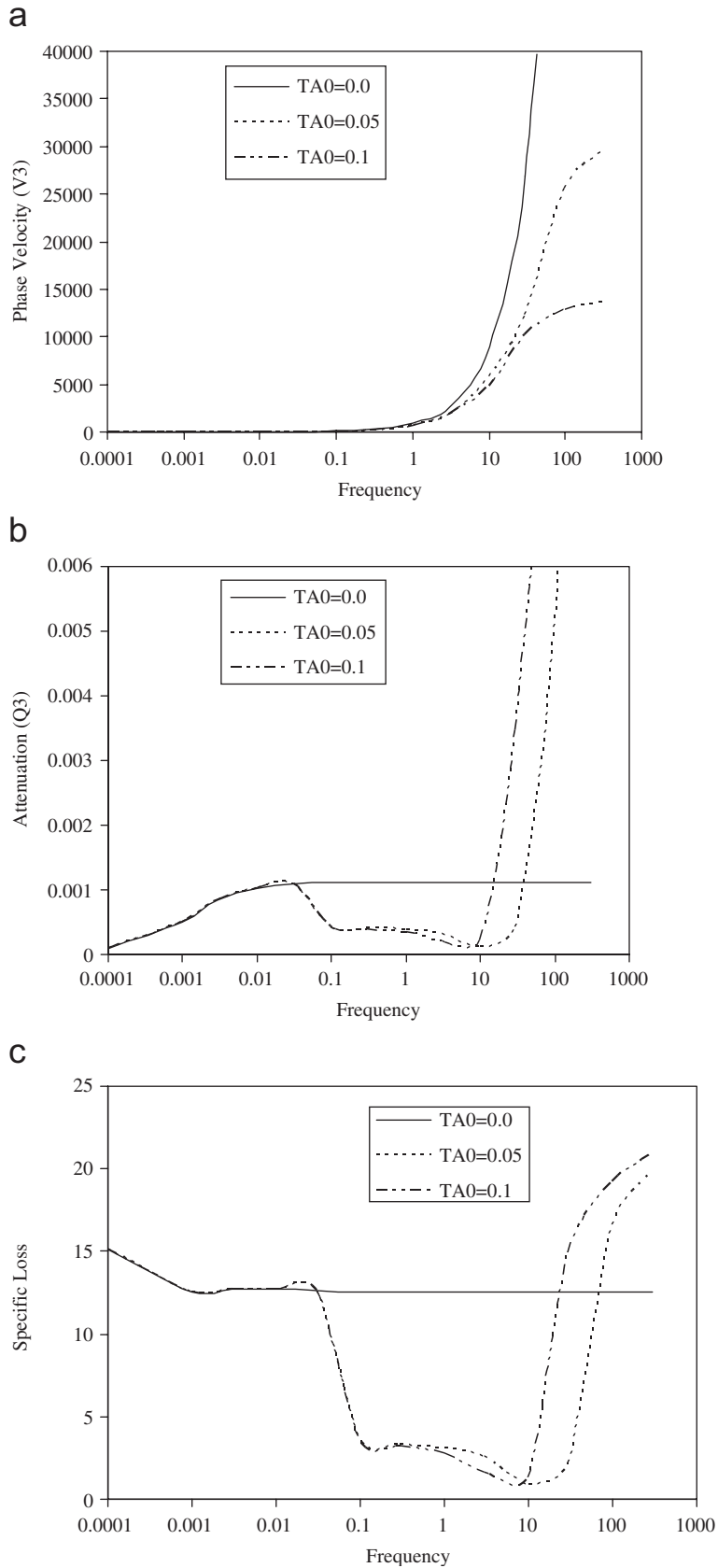
In order to illustrate and verify the analytical results obtained in the previous sections we present some numerical simulation results. The materials chosen for this purpose are copper (solvent) and zinc (solute), whose physical data is given below.

$$\begin{aligned} \lambda &= 8.2 \times 10^{10} \text{ Nm}^{-2}, \quad \mu = 4.2 \times 10^{10} \text{ Nm}^{-2}, \quad \rho = 8.950 \times 10^3 \text{ kg m}^{-3}, \quad T_0 = 300 \text{ K} \\ C_e &= 0.8298 \times 10^{-3} \text{ J/kg/K}, \quad K = 1.13 \times 10^2 \text{ W/m/s/K}, \\ \alpha_T &= 1.0 \times 10^{-8} \text{ K}^{-1}, \quad D = 0.34 \times 10^{-4} \text{ ms}^{-1}(\text{zn-cu}), \quad \epsilon_T = 0.00265, \\ \beta_1 &= 3300 \text{ Nm}^{-2} \text{ K}^{-1}, \quad \beta_2 = 330 \text{ Nm}^{-5} \text{ K}^{-1}, \quad \omega^* = 1.11 \times 10^{11} \text{ s}^{-1}, \\ \alpha_c &= 1.0 \times 10^{-9} \text{ K}^{-1}, \quad a = 0.1521 \times 10^2 \text{ ms}^{-1}, \quad b = 0.02 \times 10^{-4} \text{ ms}^{-1}, \end{aligned}$$

The secular equation (26) is solved by Cardano’s algorithm by taking $t_0 = 0.0, 0.05, 0.1$ and $t_1 = 0.0, 0.04, 0.09$ for various values of non-dimensional frequency $\omega(\omega = 0.0, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1.0, 3.0, 10., 30., 100., 300., 1000., 3000.)$ on PENTIUM-IV IBM processor by developing a FORTRAN code in order to illustrate the analytical development numerically. The values of thermal relaxation time t_0 have been estimated from Eq. (2.5) of Ref. [4] and those for t_1 are taken proportionally. The computed values of phase velocities (V_i), attenuation coefficients (Q_i) and specific loss (S_i) of three possible modes of wave propagation namely; elastodiffusive (ED) wave, thermodiffusive (TD) wave, and mass diffusion (MD) wave, are plotted in Figs. 1, 2, and 3 respectively, with frequency on log-linear scales.

From Fig. 1a, it is noticed that the phase velocity of elastodiffusive (ED) wave remains close to its isentropic value and varies linearly in the frequency range $0 \leq \omega \leq 0.03$ for all the considered values of thermal relaxation time. This quantity varies monotonically with increasing frequency in $0.03 \leq \omega \leq 1.0$ for $t_0 = 0.0, 0.05,$ and 0.1 . The phase velocity of ED-wave observes sharp increase in the range $1.0 \leq \omega \leq 10$ for all the considered values of the relaxation time. This approach to an indefinitely large value in case of $t_0 = 0.0$ but tends to finite stable values after observing slight decrease for $t_0 = 0.05$ and at $\omega \geq 10$. However, the magnitude of phase velocity is observed to fall significantly with increasing values of the thermal relaxation time t_0 in the high-frequency

Fig. 2. (a) Variation of phase velocity of thermodiffusive (TD) waves w.r.t. frequency for different relaxation times. (b) Variation of attenuation coefficient of thermodiffusive (TD) waves w.r.t. frequency for different relaxation times. (c) Variation of specific loss factor of thermodiffusive (TD) waves w.r.t. frequency for different relaxation times.



range $\omega \geq 10$. It is also noticed from the phase velocity profiles that the mean of energy transportation is mechanical in the low-frequency region and predominantly diffusion at high-frequency one.

The variations of phase velocity of mass diffusion (MD) wave are presented in Fig. 2a with increasing frequency for different values of thermal relaxation time. The trend of variations of this quantity are similar to those of Fig. 1a except that the magnitude is almost double in the instant case and the profiles pertaining to $t_0 = 0.0, 0.05$, and 0.1 have some common points in the frequency range of $1.0 \leq \omega \leq 30$. Moreover, the sharp rise in the magnitude of phase velocity in case of $t_0 = 0.0$ occurs in the frequency range $\omega \geq 10.0$ here instead of $\omega \geq 1.0$. The Fig. 3a represents phase velocity profiles of thermodiffusive (TD) waves. The trends of variation of this quantity in this case are almost similar except that the magnitude is 1.5 times more than that in Fig. 1a. The phase velocity is finite, though quite large, with increasing trend for $\omega \geq 1.0$ even in case of $t_0 = 0.05$ and 0.1 . Figs. 1a, 2a and 3a reveal that the phase velocities of all the modes namely ED, MD and TD are very close to their isentropic values in the low-frequency regime $\omega \geq 1.0$, because the energy transportation is predominantly took place through a mechanical/displacement waves in this range of frequency. At high frequency, the phase velocities of these waves shoots up in order to have large values because the mean of energy transportation for $\omega > 1$ is predominantly either thermal diffusion or mass diffusion or combination of both in this case.

The attenuation coefficient profiles of ED, MD and TD waves are presented in Figs. 1b, 2b and 3b respectively, on log-linear scales with respect to frequency for different values of relaxation time. From Fig. 1b, it is observed that attenuation coefficient (Q_1) is zero in the frequency range $0.0 \leq \omega \leq 0.001$. It sharply increases in the range $0.001 \leq \omega \leq 0.03$, decreases for $0.03 \leq \omega \leq 0.01$ and remains constant in the region $0.01 \leq \omega \leq 3.0$ for the considered values of thermal relaxation time $t_0 = 0.0, 0.05$, and 0.1 . While the variations of attenuation coefficient (Q_1) in case of coupled thermoelasticity ($t_0 = 0.0$) remains constant and vary linearly for all $\omega \geq 0.01$, it is quite dispersive in generalized thermoelasticity ($t_0 = 0.05, 0.1$) in the frequency range $3.0 \leq \omega \leq 100$ before it becomes stable and steady for $\omega \geq 100$. The attenuation coefficient (Q_2) of MD-wave is presented in Fig. 2b. It follows Gaussian distribution about $\omega = 0.001$ in the frequency range $0 < \omega \leq 0.03$ for all the considered values of thermal relaxation time. This quantity increases slowly in the frequency range $0.03 < \omega \leq 0.01$ to becomes stable and it varies approximately in linear manner afterwards for $\omega \geq 0.01$ in the context of coupled thermoelasticity ($t_0 = 0.0$). The attenuation coefficient (Q_2) of mass diffusion (MD) wave increases steadily in the frequency range $0.03 \leq \omega \leq 0.01$ and sharply afterwards for $\omega \geq 0.01$ in case of generalized thermoelasticity ($t_0 = 0.05, 0.1$). This quantity also has quite large magnitude in this range of frequency in the context of generalized thermoelasticity as compared to that in coupled thermoelasticity. It is noticed from Fig. 3b that the attenuation coefficient (Q_3) of thermodiffusive (TD) wave increases slowly in the frequency range $0 < \omega \leq 0.03$ in case of coupled ($t_0 = 0.0$) and generalized ($t_0 = 0.05, 0.1$) theories of thermoelasticity. While this quantity decreases in case of generalized thermoelasticity ($t_0 = 0.05, 0.1$) in the frequency range of $0.3 \leq \omega \leq 10$ and sharply increases for all $\omega \geq 10$, but it increases in a linear manner for $\omega \geq 0.3$ in coupled thermoelasticity ($t_0 = 0.0$). From Figs. 2b and 3b, it is observed that the attenuation coefficients of MD and TD waves start increasing from non-zero values at $\omega = 0.0001$ in contrast to the attenuation coefficient of ED—wave which remains zero even up to $\omega = 0.001$ and observes sharp increase afterwards. This agrees with the analytic results, although the development of attenuations (Q_1, Q_2, Q_3) is quite small in magnitude.

The Figs. 1c, 2c and 3c represent variation profiles of specific loss of energy in case of elastodiffusive (ED), mass diffusion (MD) and thermodiffusive (TD) waves on log-linear scales with frequency respectively. The trends of variation of ED-waves are similar to that of attenuation in the range of $0 < \omega \leq 0.1$ except having quite high magnitude as observed from Fig. 1c. It remains stable and approximately linear in coupled thermoelasticity ($t_0 = 0.0$), but drops to zero value after being dispersive in case of generalized thermoelasticity ($t_0 = 0.05, 0.1$) for $\omega \geq 0.1$. The variation profiles of specific loss of MD-waves are more or less similar to those of attenuation (Q_2) except it has quite high magnitude and also have finite values at high

Fig. 3. (a) Variation of phase velocity of mass diffusion (MD) waves w.r.t. frequency for different relaxation times. (b) Variation of attenuation coefficient of mass diffusion (MD) waves w.r.t. frequency for different relaxation times. (c) Variation of specific loss factor of mass diffusion (MD) waves w.r.t. frequency for different relaxation times.

frequencies in case of generalized thermoelasticity ($t_0 = 0.05, 0.1$). It is noticed from Fig. 3c, that specific loss profiles of TD-wave is just the mirror image of its attenuation profiles in the low-frequency range $0 < \omega \leq 0.03$ and follows almost similar trends for $\omega \geq 0.03$ except having quite large magnitude and also becomes steady and stable in case of generalized thermoelasticity ($t_0 = 0.05, 0.1$). It is noticed that, the specific loss of energy is quite high in the frequency regimes in which phase velocity has small magnitude, which signifies the effect of internal friction of the material.

From the comparison of Figs. 1a, b, 2a, b and 3a, b, it is observed that the effect of thermal relaxation time is quite significant at high-frequency regions which established the fact that “second sound” effects are short lived. The modifications of various considered quantities at low-frequency regions are mainly due to mass concentration effects and at high frequency it is due to both mass and thermal diffusion processes. Therefore, the fact that at low-frequencies ($\omega \ll 1$) mechanical energy transfer is more effective than thermal conduction as well as mass concentration and conditions locally are isentropic. However, at the opposite extreme ($\omega \gg 1$), thermal energy transfer and mass diffusion are more predominant processes and prevailing conditions are nearly isothermal and diffusive.

8. Conclusions

This work deals with the propagation of plane harmonic generalized thermoelastic diffusive waves in heat conducting solids. The secular equation infers that there are three longitudinal waves namely, a elastodiffusive (ED-mode), a mass diffusion (MD-mode) and a thermodiffusive (TD-mode), in addition to two transverse waves which can propagate in such solids. The transverse waves get decoupled from rest of the motion and hence do not interact with thermal and mass diffusion fields. These waves travel without attenuation and dispersion while other are significantly influenced by the interacting fields and hence are subjected to both attenuation and dispersion. At low frequency, the disturbance is dominated by mechanical process of transportation of energy while at high frequency it is significantly dominated by a close diffusive process (heat conduction and mass diffusion). The complex secular equation is solved both analytically and numerically by using irreducible case of Cardano’s method with the help of DeMoivre’s theorem in order to obtain phase velocities, attenuation coefficients, and specific loss of energy dissipation of ED, MD, and TD waves. It is noticed from the numerically computer simulated results that the effect of thermal relaxation time is quite significant at high frequency regions and establishing the fact that ‘second sound’ effects are short lived. The modifications of various considered quantities at low frequency are mainly due to mass concentration and at high frequency; it happens due to both mass concentration and thermal diffusion fields. Again this also strengthens the fact that mechanical energy transfer is more dominant at low frequency conditions in contrast to mass diffusion and thermal energy transfer at high frequency. Moreover, this is well exhibited and established from the various graphical representation of considered functions that, the magnitude of attenuation coefficient of all the possible modes of wave propagation is quite small as compared to respective phase velocities and specific loss factors for the considered material parameters. This shows that the interaction of mass concentration and thermal fields with displacement field enhances the signal propagation to distant positions as compared to that in its absence.

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References

- [1] H.W. Lord, Y. Shulman, The generalized dynamical theory of thermoelasticity, *Journal of Mechanics and Physics of Solids* 15 (1967) 299–309.
- [2] A.E. Green, K.A. Lindsay, Thermoelasticity, *Journal of Elasticity* 2 (1972) 1–7.
- [3] R.S. Dhaliwal, H.H. Sherief, Generalized thermoelasticity for anisotropic media, *Quarterly of Applied Mathematics* 33 (1980) 1–8.
- [4] D.S. Chandrashekhariah, Thermoelasticity with second: a review, *Applied Mechanics Review* 39 (1986) 355–376.

- [5] R.B. Hetnarski, J. Ignaczak, Generalized thermoelasticity, *Journal of Thermal Stresses* 22 (1999) 451–476.
- [6] W. Nowacki, Dynamical problems of thermodiffusion in solids-I, *Bulletin of Polish Academy of Sciences Series, Science and Technology* 22 (1974) 55–64.
- [7] W. Nowacki, Dynamic problems of thermodiffusion in solids-II, *Bulletin of Polish Academy of Sciences Series, Science and Technology* 22 (1974) 129–135.
- [8] W. Nowacki, Dynamic problems of thermodiffusion in solids-III, *Bulletin of Polish Academy of Sciences Series, Science and Technology* 22 (1974) 275–276.
- [9] W. Nowacki, Dynamic problems of thermodiffusion in solids, *Engineering Fracture Mechanics* 8 (1976) 261–266.
- [10] W. Dudziak, S.J. Kowalski, Theory of thermodiffusion for solids, *International Journal of Heat and Mass Transfer* 32 (1989) 2005–2013.
- [11] Z.S. Olesiak, Y.A. Pyryev, A coupled quasi-stationary problem of thermodiffusion for an elastic cylinder, *International Journal of Engineering Science* 33 (1995) 773–780.
- [12] H.H. Sherief, F. Hamza, H. Saleh, The theory of generalized thermoelastic diffusion, *International Journal of Engineering Science* 42 (2004) 591–608.
- [13] J.N. Sharma, H. Singh, Generalized thermoelastic waves in anisotropic media, *Journal of Acoustical Society of America* 85 (1989) 1407–1413.
- [14] J.N. Sharma, V. Kumar, S.P. Sud, Plane harmonic waves in orthotropic thermoelastic materials, *Journal of Acoustical Society of America* 107 (2000) 293–305.
- [15] J.N. Sharma, M. Pal, Plane harmonic waves in transversely isotropic piezothermoelastic materials, *International Journal of Applied Mechanics and Engineering* 9 (2004) 749–769.
- [16] F. Ahmad, A. Khan, Thermoelastic plane waves in a rotating isotropic medium, *Acta Mechanica* 136 (1999) 243–247.
- [17] B. Singh, Reflection of P and SV waves from free surface of an elastic solid with generalized thermodiffusion, *Journal of Earth and System Sciences* 114 (2005) 159–168.
- [18] P. Puri, S.C. Cowin, Plane waves in linear elastic materials with voids, *Journal of Elasticity* 15 (1985) 167–183.
- [19] H. Kolsky, *Stress Waves in Solids*, Clarendon Press, Oxford, 1935 Dover Press New York, 1963.